

AN ANALOGUE OF FEKETE'S LEMMA FOR SUBADDITIVE FUNCTIONS ON CANCELLATIVE AMENABLE SEMIGROUPS

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ABSTRACT. We prove an analogue of Fekete's lemma for subadditive right-subinvariant functions defined on the finite subsets of a cancellative left-amenable semigroup. This extends results previously obtained in the case of amenable groups by E. Lindenstrauss and B. Weiss and by M. Gromov.

1. INTRODUCTION

Fekete's lemma [7] is a classical result in undergraduate-level analysis. It states that if $(u_n)_{n \geq 1}$ is a subadditive sequence of real numbers then the sequence $\left(\frac{u_n}{n}\right)_{n \geq 1}$ has a limit $\lambda \in \mathbb{R} \cup \{-\infty\}$ as n tends to infinity (see for example [12, Proposition 9.6.4]). The goal of the present paper is to give an analogue of Fekete's lemma for subadditive and right-subinvariant functions defined on the set of all finite subsets of a cancellative left-amenable semigroup. In order to state our main result, let us first recall some basic definitions and introduce notation.

Let S be a semigroup, i.e., a set equipped with an associative binary operation. We denote by $\mathcal{P}(S)$ the set of all subsets of S . One says that S is *left-amenable* if there exists a finitely additive left-invariant probability measure defined on $\mathcal{P}(S)$, that is, a map $\mu: \mathcal{P}(S) \rightarrow [0, 1]$ satisfying the following conditions:

- (A1) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathcal{P}(S)$ such that $A \cap B = \emptyset$;
- (A2) $\mu(S) = 1$;
- (A3) $\mu(L_s^{-1}(A)) = \mu(A)$ for all $s \in S$ and $A \in \mathcal{P}(S)$,

where $L_s: S \rightarrow S$ denotes the left-multiplication by s , that is, the map defined by $L_s(t) = st$ for all $t \in S$.

One says that S is *right-amenable* if its opposite semigroup is left-amenable. This is equivalent to the existence of a finitely additive right-invariant probability measure defined on $\mathcal{P}(S)$, that is, a map $\mu: \mathcal{P}(S) \rightarrow [0, 1]$ satisfying (A1), (A2) and

- (A3') $\mu(R_s^{-1}(A)) = \mu(A)$ for all $s \in S$ and $A \in \mathcal{P}(S)$,

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where $R_s: S \rightarrow S$ is the right-multiplication by s , that is, the map defined by $R_s(t) = ts$ for all $t \in S$.

A semigroup is called *amenable* if it is both left-amenable and right-amenable.

The notion of an amenable group was introduced in 1929 by J. von Neumann [17]. His original motivation was the study of the Banach-Tarski paradox. The theory of amenable semigroups was subsequently developed in the 1940s and 1950s by M. Day (see [4], [5], [6], and the references therein). Day [4] showed in particular that every commutative semigroup is amenable, thus extending a result previously obtained by von Neumann for groups. Actually, when passing from groups to semigroups, one encounters many new phenomena. For example, left-amenability and right-amenability are equivalent for groups, every finite group is amenable, and every subgroup of an amenable group is itself amenable. On the other hand, in contrast with the group case, there exist finite semigroups that are left-amenable but not right-amenable and finite amenable semigroups containing semigroups that are neither left-amenable nor right-amenable.

In the group setting, E. Følner [8] gave a remarkable combinatorial characterization of amenability by showing that a group S is amenable if and only if it satisfies the following condition:

(FC) for every finite subset $K \subset S$ and every real number $\varepsilon > 0$, there exists a non-empty finite subset $F \subset S$ such that

$$(1.1) \quad |kF \setminus F| \leq \varepsilon|F| \quad \text{for all } k \in K.$$

(Here and in the sequel, we use $|\cdot|$ to denote cardinality of finite sets.) Condition (FC) is known as the *Følner condition*.

In his thesis, A. Frey [9] adapted Følner arguments to semigroups and showed that every left-amenable semigroup S satisfies condition (FC) (see [16, Theorem 3.5] for a simpler proof). However, (FC) is a necessary but not sufficient condition for left-amenability of semigroups. Examples of semigroups that are not left-amenable but satisfy (FC) are provided by finite semigroups that are not left-amenable (observe that any finite semigroup S trivially satisfies (FC) by taking $F = S$).

Condition (FC) is equivalent to the existence of a directed net $(F_i)_{i \in I}$ of non-empty finite subsets of S such that

$$(1.2) \quad \lim_i \frac{|sF_i \setminus F_i|}{|F_i|} = 0 \quad \text{for all } s \in S.$$

Indeed, if S satisfies (FC), we can construct a directed net $(F_i)_{i \in I}$ satisfying (1.2) in the following way. We first take as I the directed set consisting of all pairs (K, ε) , where K is a finite subset of S and $\varepsilon > 0$, with the partial ordering on I defined by $(K_1, \varepsilon_1) \leq (K_2, \varepsilon_2)$ if and only if $K_1 \subset K_2$ and $\varepsilon_2 \leq \varepsilon_1$. Then, for each $i = (K, \varepsilon) \in I$, we take as F_i one of the non-empty finite subsets $F \subset S$ satisfying (1.1). Conversely, suppose that $(F_i)_{i \in I}$ is a directed net of non-empty finite subsets of S satisfying (1.2). Let $K \subset S$ be a finite subset and $\varepsilon > 0$. Then, for every $k \in K$, we can find $i_k \in I$ such that $|kF_{i_k} \setminus F_{i_k}| \leq \varepsilon|F_{i_k}|$ for all $i \geq i_k$. To get a non-empty finite subset $F \subset S$ satisfying (1.1), it suffices to take $F := F_i$,

where $i \in I$ is such that $i \geq i_k$ for every $k \in K$ (the existence of such an index i follows from the fact that I is directed and K is finite).

A directed net $(F_i)_{i \in I}$ of non-empty finite subsets of S satisfying (1.2) is called a *left-Følner net* of S .

Recall that an element s in a semigroup S is called *left-cancellable* (resp. *right-cancellable*) if the map L_s (resp. R_s) is injective. One says that s is cancellable if it is both left-cancellable and right-cancellable. The semigroup S is called *left-cancellative* (resp. *right-cancellative*, resp. *cancellative*) if every element in S is left-cancellable (resp. right-cancellable, resp. cancellable).

When S is a left-cancellative semigroup, it is known that the left-amenability of S is equivalent to the Følner condition (FC), and hence to the existence of a left-Følner net (see [16, Corollary 4.3]).

The purpose of the present paper is to establish the following result.

Theorem 1.1. *Let S be a cancellative left-amenable semigroup and let $\mathcal{P}_{fin}(S)$ denote the set of all finite subsets of S . Let $h: \mathcal{P}_{fin}(S) \rightarrow \mathbb{R}$ be a real-valued map satisfying the following conditions:*

(H1) *h is subadditive, i.e.,*

$$h(A \cup B) \leq h(A) + h(B) \text{ for all } A, B \in \mathcal{P}_{fin}(S);$$

(H2) *h is right-subinvariant, i.e.,*

$$h(As) \leq h(A) \text{ for all } s \in S \text{ and } A \in \mathcal{P}_{fin}(S);$$

(H3) *h is bounded on singletons, i.e., there exists a real number $M \geq 0$ such that*

$$h(\{s\}) \leq M \text{ for all } s \in S.$$

Then there exists a real number $\lambda \geq 0$, depending only on h , such that the net $\left(\frac{h(F_i)}{|F_i|}\right)_{i \in I}$ converges to λ for every left-Følner net $(F_i)_{i \in I}$ of S .

Observe that conditions (H3) is implied by (H2) when S admits an element s_0 such that the map L_{s_0} is onto. Indeed, in this case, (H2) gives us $h(\{s\}) \leq h(\{s_0\})$ for all $s \in S$. This happens for example when S is a monoid (i.e., S admits an identity element) since we can then take $s_0 = 1_S$. Thus, an immediate consequence of Theorem 1.1 is the following result.

Corollary 1.2. *Let S be a cancellative left-amenable monoid and let $h: \mathcal{P}_{fin}(S) \rightarrow \mathbb{R}$ be a subadditive and right-subinvariant map. Then there exists a real number $\lambda \geq 0$, depending only on h , such that the net $\left(\frac{h(F_i)}{|F_i|}\right)_{i \in I}$ converges to λ for every left Følner net $(F_i)_{i \in I}$ of S .*

As far as we know, the above result is new even in the case when S is the additive monoid \mathbb{N} of non-negative integers. However, when S is an amenable group, it was previously established by E. Lindenstrauss and B. Weiss [15, Theorem 6.1] under the additional

assumption that h is non-decreasing ($h(A) \leq h(B)$ for all $A, B \in \mathcal{P}_{fin}(S)$ such that $A \subset B$), and by M. Gromov [10, Section 1.3.1].

Let us note that, in the case when S is a group, condition (H2) implies that h is right-invariant since we then have $h(A) = h(Ass^{-1}) \leq h(As)$ and hence $h(As) = h(A)$ for all $s \in S$ and $A \in \mathcal{P}_{fin}(S)$. This is no more true in general for semigroups. For example, if S is the additive monoid \mathbb{N} , then the map $h: \mathcal{P}_{fin}(\mathbb{N}) \rightarrow \mathbb{R}$ defined by $h(A) = (1 + \max(A))^{-1}|A|$ clearly satisfies (H1) and (H2) but $h(A + s) < h(A)$ for $A = \{0\}$ and $s = 1$.

The proof of Lindenstrauss and Weiss is based on the Ornstein-Weiss machinery of quasi-tiles (cf. [18, Section I.2: Theorem 6]). This is the reason why the group version of Theorem 1.1 is sometimes called the *Ornstein-Weiss lemma* although it does not appear explicitly in [18]. A detailed exposition of Gromov's proof of the Ornstein-Weiss lemma may be found in [14].

In the theory of dynamical systems, Theorem 1.1 is important in defining numerical invariants such as topological entropy, measure-theoretic entropy, and mean topological dimension. These invariants are obtained by taking limits of quantities defined from a left-Følner net and one can deduce from Theorem 1.1 that the choice of the left-Følner net is actually irrelevant for actions of cancellative left-amenable semigroups.

Our proof of Theorem 1.1 is entirely self-contained. At several points, it is inspired by some of the ideas developed by Gromov in [10, Section 1.3.1]. However, here again, the passage from groups to semigroups inevitably imposes significant modifications in the arguments.

We do not know to what extent Theorem 1.1 remains valid for non-cancellative left-amenable semigroups.

The paper is organized as follows. In Section 2, we establish some general properties of boundary sets in semigroups that are needed for the proof of our main result. We give in particular a characterization of Følner nets for cancellative semigroups in terms of relative amenability. In Section 3, we introduce a notion of ε -filling pattern for finite subsets of semigroups and prove a theorem about the existence of certain fillings by finite systems of tiles with small relative amenability (Theorem 3.8). This filling theorem is a key tool in the proof of Theorem 1.1 given in Section 4. In Section 5, which is mostly expository, we discuss the above-mentioned applications of Theorem 1.1 to the definition of numerical invariants of dynamical systems.

2. BOUNDARIES AND RELATIVE AMENABILITY

Let S be a semigroup. Let K and A be subsets of S .

The *right K -interior* of A is the set

$$\text{Int}_K(A) := \{s \in A : Ks \subset A\}$$

consisting of all the elements s in A such that the right-translate of K by s is entirely contained in A .

The *right K -boundary* of A is the set $\partial_K(A) \subset A$ defined by

$$\partial_K(A) := A \setminus \text{Int}_K(A).$$

Thus, an element $s \in S$ is in $\partial_K(A)$ if and only if s is in A and Ks meets the complement of A in S .

Proposition 2.1. *Let S be a semigroup. Let K , A , and B be subsets of S . Then one has*

- (i) $\partial_K(A) = A \cap (\bigcup_{k \in K} L_k^{-1}(kA \setminus A))$;
- (ii) *if every element in K is left-cancellable then $\partial_K(A) = \bigcup_{k \in K} L_k^{-1}(kA \setminus A)$;*
- (iii) $\partial_K(A \cup B) \subset \partial_K(A) \cup \partial_K(B)$;
- (iv) $\partial_K(B \setminus A) \subset (\partial_K(B) \cup (\bigcup_{k \in K} L_k^{-1}(A \cap kS))) \setminus A$;
- (v) *if s is a right-cancellable element of S then*

$$(\text{Int}_K(A))s = \text{Int}_K(As);$$

- (vi) *if s is a right-cancellable element of S then*

$$(\partial_K(A))s = \partial_K(As).$$

Proof. (i) This is clear since $s \in \partial_K(A)$ if and only if $s \in A$ and $ks \notin A$ for some $k \in K$.

(ii) This immediately follows from (i) since the injectivity of L_k implies that $L_k^{-1}(kA \setminus A) \subset A$.

(iii) Let $s \in \partial_K(A \cup B)$. This means that $s \in A \cup B$ and

$$Ks \cap (S \setminus (A \cup B)) \neq \emptyset.$$

Since $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$, we deduce that $s \in \partial_K(A) \cup \partial_K(B)$.

(iv) Suppose that $s \in \partial_K(B \setminus A)$. This means that $s \in B \setminus A$ and

$$Ks \cap (S \setminus (B \setminus A)) \neq \emptyset.$$

Since $S \setminus (B \setminus A) = (S \setminus B) \cup A$, we deduce that if $s \notin \partial_K(B)$, then $Ks \cap A \neq \emptyset$ and hence $s \in \bigcup_{k \in K} L_k^{-1}(A \cap kS)$. As $s \notin A$, inclusion (iv) immediately follows.

(v) Suppose that $s \in S$ is right-cancellable and let $g \in (\text{Int}_K(A))s$. This means that there exists $a \in \text{Int}_K(A)$ such that $g = as$. Hence $g \in As$ and $Kg = K(as) = (Ka)s \subset As$ since $a \in \text{Int}_K(A)$. Thus $g \in \text{Int}_K(As)$. This gives the inclusion $(\text{Int}_K(A))s \subset \text{Int}_K(As)$.

Conversely, suppose now that $g \in \text{Int}_K(As)$. Then $g \in As$ and $Kg \subset As$. Thus, there exists $a \in A$ such that $g = as$ and $(Ka)s = K(as) \subset As$. Remark that the inclusion $(Ka)s \subset As$ is equivalent to the inclusion $Ka \subset A$ by injectivity of R_s . This proves that $a \in \text{Int}_K(A)$ so that $g \in (\text{Int}_K(A))s$. Hence $\text{Int}_K(As) \subset (\text{Int}_K(A))s$. This completes the proof of (v).

(vi) If $s \in S$ is right-cancellable, we have

$$\begin{aligned} (\partial_K(A))s &= (A \setminus \text{Int}_K(A))s \\ &= As \setminus (\text{Int}_K(A))s && \text{(since } R_s \text{ is injective)} \\ &= As \setminus \text{Int}_K(As) && \text{(by (v))} \\ &= \partial_K(As). \end{aligned}$$

This shows (vi). □

Lemma 2.2. *Let S be a semigroup. Suppose that K and A are finite subsets of S and that every element of K is left-cancellable. Then one has*

$$(2.1) \quad |\partial_K(A)| \leq \sum_{k \in K} |kA \setminus A|$$

and

$$(2.2) \quad |kA \setminus A| \leq |\partial_K(A)| \quad \text{for all } k \in K.$$

Proof. It follows from Proposition 2.1.(ii) that

$$(2.3) \quad \partial_K(A) = \bigcup_{k \in K} L_k^{-1}(kA \setminus A).$$

This implies

$$|\partial_K(A)| = \left| \bigcup_{k \in K} L_k^{-1}(kA \setminus A) \right| \leq \sum_{k \in K} |L_k^{-1}(kA \setminus A)|.$$

As $|L_k^{-1}(kA \setminus A)| = |kA \setminus A|$ for all $k \in K$ by injectivity of L_k , this gives us (2.1).

On the other hand, given $k \in K$, we deduce from (2.3) that

$$L_k^{-1}(kA \setminus A) \subset \partial_K(A).$$

This implies

$$|kA \setminus A| = |L_k^{-1}(kA \setminus A)| \leq |\partial_K(A)|$$

which yields (2.2). □

Let A and K be subsets of S with A finite and non-empty. Then $\partial_K(A)$ is also finite since $\partial_K(A) \subset A$. We define the *amenability constant* of A with respect to K by

$$\alpha(A, K) := \frac{|\partial_K(A)|}{|A|}.$$

Note that $\alpha(A, K)$ is rational and that one has $0 \leq \alpha(A, K) \leq 1$.

For left-cancellative semigroups, left-amenability is equivalent to the existence of finite subsets with arbitrary small relative amenability. More precisely, we have the following result.

Proposition 2.3. *Let S be a left-cancellative semigroup. Then the following conditions are equivalent:*

- (a) S is left-amenable;
- (b) for every finite subset K of S and every real number $\varepsilon > 0$, there exists a non-empty finite subset F of S such that $\alpha(F, K) \leq \varepsilon$.

Proof. Let F and K be finite subsets of S with $F \neq \emptyset$. From inequality (2.1) of Lemma 2.2, we deduce that if $|kF \setminus F| \leq \varepsilon|F|$ for all $k \in K$, then $\alpha(F, K) \leq |K|\varepsilon$. Conversely, inequality (2.2) implies that if $\alpha(F, K) \leq \varepsilon$ then $|kF \setminus F| \leq \varepsilon|F|$ for all $k \in K$.

We deduce that the Følner condition (FC) is equivalent to condition (b) of the statement. On the other hand, as S is left-cancellative, we know from the result mentioned

in the Introduction that S is left-amenable if and only if it satisfies (FC). This shows the equivalence between conditions (a) and (b). \square

Similarly, we have the following characterization of Følner nets in left-cancellative and left-amenable semigroups.

Proposition 2.4. *Let S be a left-cancellative and left-amenable semigroup. Let $(F_i)_{i \in I}$ be a directed net of non-empty finite subsets of S . Then the following conditions are equivalent:*

- (a) $(F_i)_{i \in I}$ is a left-Følner net for S ;
- (b) for each finite subset K of S , one has $\lim_i \alpha(F_i, K) = 0$.

Proof. Let $s \in S$ and take $K = \{s\}$. Then one has $|sF_i \setminus F_i|/|F_i| \leq \alpha(F_i, K)$ for all $i \in I$ by (2.2). This shows that (b) implies (a).

Conversely, suppose that $(F_i)_{i \in I}$ is a left-Følner net for S . Let K be a finite subset of S and $\varepsilon > 0$. Then there exists $i_k \in I$ such that $|kF_i \setminus F_i|/|F_i| \leq \varepsilon$ for all $i \geq i_k$. If $j \in I$ is such that $j \geq i_k$ for all $k \in K$, we deduce that $\alpha(F_i, K) \leq \varepsilon|K|$ for all $i \geq j$ by using (2.1). This shows that (a) implies (b). \square

3. FILLINGS

The goal of this section is to establish Theorem 3.8 which is a key tool in the proof of Theorem 1.1 that will be given in the next section.

Definition 3.1. Let X be a set and $\varepsilon > 0$ a real number. A family $(A_j)_{j \in J}$ of finite subsets of X is said to be ε -disjoint if there exists a family $(B_j)_{j \in J}$ of pairwise disjoint subsets of X such that $B_j \subset A_j$ and $|B_j| \geq (1 - \varepsilon)|A_j|$ for all $j \in J$.

Lemma 3.2. *Let X be a set and $(A_j)_{j \in J}$ a finite ε -disjoint family of finite subsets of X . Then one has*

$$(1 - \varepsilon) \sum_{j \in J} |A_j| \leq \left| \bigcup_{j \in J} A_j \right|.$$

Proof. Since $(A_j)_{j \in J}$ is ε -disjoint, there exists a family $(B_j)_{j \in J}$ of pairwise disjoint subsets of X such that $B_j \subset A_j$ and $|B_j| \geq (1 - \varepsilon)|A_j|$ for all $j \in J$. Thus, we have

$$(1 - \varepsilon) \sum_{j \in J} |A_j| \leq \sum_{j \in J} |B_j| = \left| \bigcup_{j \in J} B_j \right| \leq \left| \bigcup_{j \in J} A_j \right|.$$

\square

Lemma 3.3. *Let S be a semigroup. Let also K be a finite subset of S and $0 < \varepsilon < 1$. Suppose that $(A_j)_{j \in J}$ is a finite ε -disjoint family of non-empty finite subsets of S . Then one has*

$$\alpha\left(\bigcup_{j \in J} A_j, K\right) \leq \frac{1}{1 - \varepsilon} \max_{j \in J} \alpha(A_j, K).$$

Proof. Let us set $M := \max_{j \in J} \alpha(A_j, K)$. It follows from Proposition 2.1.(iii) that

$$\partial_K \left(\bigcup_{j \in J} A_j \right) \subset \bigcup_{j \in J} \partial_K(A_j).$$

Thus

$$\left| \partial_K \left(\bigcup_{j \in J} A_j \right) \right| \leq \left| \bigcup_{j \in J} \partial_K(A_j) \right| \leq \sum_{j \in J} |\partial_K(A_j)| = \sum_{j \in J} \alpha(A_j, K) |A_j| \leq M \sum_{j \in J} |A_j|.$$

As the family $(A_j)_{j \in J}$ is ε -disjoint, we deduce from Lemma 3.2 that

$$\alpha \left(\bigcup_{j \in J} A_j, K \right) = \frac{\left| \partial_K \left(\bigcup_{j \in J} A_j \right) \right|}{\left| \bigcup_{j \in J} A_j \right|} \leq \frac{M}{1 - \varepsilon}.$$

□

Lemma 3.4. *Let S be a semigroup. Let K , A and Ω be finite subsets of S such that every element of K is left-cancellable and $\emptyset \neq A \subset \Omega$. Suppose that $\varepsilon > 0$ is a real number such that $|\Omega \setminus A| \geq \varepsilon |\Omega|$. Then one has*

$$\alpha(\Omega \setminus A, K) \leq \frac{\alpha(\Omega, K) + |K| \alpha(A, K)}{\varepsilon}.$$

Proof. By Proposition 2.1.(iv), we have that

$$\partial_K(\Omega \setminus A) \subset \left(\partial_K(\Omega) \cup \left(\bigcup_{k \in K} L_k^{-1}(A \cap kS) \right) \right) \setminus A.$$

This implies

$$\partial_K(\Omega \setminus A) \subset \partial_K(\Omega) \cup \left(\bigcup_{k \in K} L_k^{-1}(A \cap kS) \setminus A \right)$$

and hence

$$(3.1) \quad |\partial_K(\Omega \setminus A)| \leq |\partial_K(\Omega)| + \sum_{k \in K} |L_k^{-1}(A \cap kS) \setminus A|.$$

Now, for all $k \in K$, the injectivity of L_k implies that

$$|L_k^{-1}(A \cap kS) \setminus A| = |k(L_k^{-1}(A \cap kS) \setminus A)| = |A \cap kS \setminus kA| \leq |A \setminus kA| = |kA \setminus A|,$$

where the last equality follows from the fact that A and kA have the same cardinality. Hence, by using inequality (2.2) in Lemma 2.2, we get

$$(3.2) \quad |L_k^{-1}(A \cap kS) \setminus A| \leq |\partial_K(A)|$$

for all $k \in K$.

From (3.1) and (3.2), we deduce that

$$(3.3) \quad |\partial_K(\Omega \setminus A)| \leq |\partial_K(\Omega)| + |K| |\partial_K(A)|.$$

It follows that

$$\begin{aligned}
\alpha(\Omega \setminus A, K) &= \frac{|\partial_K(\Omega \setminus A)|}{|\Omega \setminus A|} \\
&\leq \frac{|\partial_K(\Omega)| + |K||\partial_K(A)|}{|\Omega \setminus A|} && \text{(by (3.3))} \\
&= \frac{\alpha(\Omega, K)|\Omega| + |K|\alpha(A, K)|A|}{|\Omega \setminus A|} \\
&\leq \frac{\alpha(\Omega, K)|\Omega| + |K|\alpha(A, K)|A|}{\varepsilon|\Omega|} && \text{(since } |\Omega \setminus A| \geq \varepsilon|\Omega|) \\
&\leq \frac{\alpha(\Omega, K) + |K|\alpha(A, K)}{\varepsilon} && \text{(since } |A| \leq |\Omega|).
\end{aligned}$$

□

Lemma 3.5. *Let S be a semigroup. Let A and B be finite subsets of S . Suppose that every element of A is left-cancellable. Then one has*

$$\sum_{s \in S} |As \cap B| \leq |A||B|.$$

Proof. For $E \subset S$, denote by $\chi_E: S \rightarrow \mathbb{R}$ the characteristic map of E , i.e., the map defined by $\chi_E(s) = 1$ if $s \in E$ and $\chi_E(s) = 0$ otherwise. We have

$$\begin{aligned}
\sum_{s \in S} |As \cap B| &= \sum_{s \in S} \sum_{s' \in S} \chi_{As \cap B}(s') \\
&= \sum_{s \in S} \sum_{s' \in S} \chi_{As}(s') \chi_B(s') \\
(3.4) \quad &= \sum_{s' \in S} \sum_{s \in S} \chi_{As}(s') \chi_B(s') \\
&= \sum_{s' \in S} \chi_B(s') \left(\sum_{s \in S} \chi_{As}(s') \right).
\end{aligned}$$

If we fix $a \in A$ and $s' \in S$, the injectivity of L_a implies that there exists at most one element $s \in S$ such that $as = s'$. It follows that, given $s' \in S$, there are at most $|A|$ elements $s \in S$ such that $s' \in As$. In other words, we have

$$\sum_{s \in S} \chi_{As}(s') \leq |A|$$

for all $s' \in S$. Thus, we deduce from (3.4) that

$$\sum_{s \in S} |As \cap B| \leq |A| \sum_{s' \in S} \chi_B(s') = |A||B|.$$

□

Remarks. 1) The argument used in the preceding proof shows that the inequality in Lemma 3.5 can be replaced by an equality if L_a is bijective for every $a \in A$ (e.g., if S is a group). In fact, when S is a group, the equality $\sum_{s \in S} |As \cap B| = |A||B|$ is obtained by taking $f = \chi_A$ and $g = \chi_B$ in the formula $\|f * g\|_1 = \|f\|_1 \|g\|_1$, valid for all $f, g \geq 0$ in the convolution Banach algebra $\ell^1(S)$.

2) The inequality in Lemma 3.5 may be strict. Consider for example the additive monoid \mathbb{N} of non-negative integers and two non-empty finite subsets $A, B \subset \mathbb{N}$ with $\max B < \min A$. Then one has $\sum_{s \in \mathbb{N}} |(A + s) \cap B| = 0$ but $|A||B| \geq 1$. Note that the additive monoid \mathbb{N} is commutative (and hence amenable) and cancellative.

3) Lemma 3.5 becomes false if we drop the hypothesis that every element of A is left-cancellable. Indeed, consider the monoid $S = \{s_0, s_1\}$, where s_0 is an identity element and $s_1 \neq s_0$ satisfies $s_1^2 = s_1$. Then, by taking $A = B = \{s_1\}$, we have $\sum_{s \in S} |As \cap B| = 2$ but $|A||B| = 1$.

Definition 3.6. Let S be a semigroup. Let K and Ω be finite subsets of S . Given a real number $\varepsilon > 0$, a finite subset $P \subset S$ is called an (ε, K) -filling pattern for Ω if the following conditions are satisfied:

- (F1) $P \subset \text{Int}_K(\Omega)$;
- (F2) the family $(Ks)_{s \in P}$ is ε -disjoint.

The following lemma will be used in the proof of Theorem 3.8 (compare with $(+)_\varepsilon$ in Section 1.3.1 of [10] in the group case). It can be viewed as a kind of analogue of Euclidean division for integers.

Lemma 3.7 (Filling lemma). *Let S be a cancellative semigroup. Let Ω and K be non-empty finite subsets of S . Then, for every $\varepsilon \in (0, 1]$, there exists an (ε, K) -filling pattern P for Ω such that*

$$(3.5) \quad |KP| \geq \varepsilon(1 - \alpha(\Omega, K))|\Omega|.$$

Proof. Let \mathcal{P} denote the set consisting of all (ε, K) -filling patterns for Ω . Observe that \mathcal{P} is not empty, since $\emptyset \in \mathcal{P}$, and that every element of \mathcal{P} has cardinality bounded above by $|\text{Int}_K(\Omega)|$, since it is contained in $\text{Int}_K(\Omega)$. Choose a pattern $P \in \mathcal{P}$ with maximal cardinality. Let us show that (3.5) is satisfied. To slightly simplify notation, let us set

$$B := KP = \bigcup_{s \in P} Ks.$$

By applying Lemma 3.5, we get

$$(3.6) \quad \sum_{s \in \text{Int}_K(\Omega)} |Ks \cap B| \leq \sum_{s \in S} |Ks \cap B| \leq |K||B|.$$

Let us prove that

$$(3.7) \quad \varepsilon|Ks| \leq |Ks \cap B| \quad \text{for all } s \in \text{Int}_K(\Omega).$$

If $s \in P$, then $Ks \cap B = Ks$ and (3.7) holds true since $\varepsilon \leq 1$. Let now $s \in \text{Int}_K(\Omega) \setminus P$ and suppose, by contradiction, that $|Ks \cap B| < \varepsilon|Ks|$. Then, we have that

$$|Ks \setminus B| = |Ks| - |Ks \cap B| > |Ks| - \varepsilon|Ks| = (1 - \varepsilon)|Ks|,$$

which implies that $P \cup \{s\}$ is an (ε, K) -filling pattern for Ω . This contradicts the maximality of the cardinality of P . This proves (3.7).

Finally, we obtain

$$\begin{aligned} \varepsilon|K||\text{Int}_K(\Omega)| &= \sum_{s \in \text{Int}_K(\Omega)} \varepsilon|K| \\ &= \sum_{s \in \text{Int}_K(\Omega)} \varepsilon|Ks| \quad (\text{since } |K| = |Ks| \text{ by right-cancellativity of } s) \\ &\leq \sum_{s \in \text{Int}_K(\Omega)} |Ks \cap B| \quad (\text{by (3.7)}) \\ &\leq |K||B| \quad (\text{by (3.6)}), \end{aligned}$$

which gives us

$$|B| \geq \varepsilon|\text{Int}_K(\Omega)|.$$

As $|\text{Int}_K(\Omega)| = |\Omega| - |\partial_K(\Omega)| = (1 - \alpha(\Omega, K))|\Omega|$, this yields (3.5). \square

Theorem 3.8 (Filling theorem). *Let S be a cancellative semigroup and let $\varepsilon \in (0, \frac{1}{2}]$. Then there exists an integer $n_0 = n_0(\varepsilon) \geq 1$ such that for each integer $n \geq n_0$ the following holds.*

If $(K_j)_{1 \leq j \leq n}$ is a finite sequence of non-empty finite subsets of S such that

$$(3.8) \quad \alpha(K_k, K_j) \leq \frac{\varepsilon^{2n}}{|K_j|} \quad \text{for all } 1 \leq j < k \leq n,$$

and D is a non-empty finite subset of S such that

$$(3.9) \quad \alpha(D, K_j) \leq \varepsilon^{2n} \quad \text{for all } 1 \leq j \leq n,$$

then there exists a finite sequence $(P_j)_{1 \leq j \leq n}$ of finite subsets of S satisfying the following conditions:

- (T1) *the set P_j is an (ε, K_j) -filling pattern of D for every $1 \leq j \leq n$;*
- (T2) *the subsets $K_j P_j \subset D$, $1 \leq j \leq n$, are pairwise disjoint;*
- (T3) *the subset $D' \subset D$ defined by*

$$D' := D \setminus \bigcup_{1 \leq j \leq n} K_j P_j$$

is such that $|D'| \leq \varepsilon|D|$.

Proof. Fix $\varepsilon \in (0, \frac{1}{2}]$ and a positive integer n . Let K_j , $1 \leq j \leq n$, and D be non-empty finite subsets of S satisfying conditions (3.8) and (3.9).

Let us first define, by induction, a finite process with at most n steps for constructing suitable finite subsets P_n, P_{n-1}, \dots, P_1 of S . We will see that these subsets have the required properties when n is large enough, i.e., for $n \geq n_0$ with $n_0 = n_0(\varepsilon)$ that will be made precise at the end of the proof.

Step 1. We set $D_0 := D$. By (3.9), we have

$$(H(1;a)) \quad \alpha(D_0, K_j) \leq \varepsilon^{2n} \text{ for all } 1 \leq j \leq n.$$

Using Lemma 3.7 with $\Omega = D_0 = D$ and $K = K_n$, we can find a finite subset $P_n \subset S$ such that

$$(H(1;b)) \quad P_n \text{ is an } (\varepsilon, K_n)\text{-filling pattern for } D_0$$

and

$$(3.10) \quad |K_n P_n| \geq \varepsilon(1 - \alpha(D, K_n))|D| \geq \varepsilon(1 - \varepsilon^{2n})|D|.$$

$$(H(1;c)) \quad \text{Setting}$$

$$D_1 := D_0 \setminus K_n P_n,$$

we deduce from (3.10) that

$$|D_1| \leq |D|(1 - \varepsilon(1 - \varepsilon^{2n})).$$

Step k . We continue this process by induction as follows. Suppose that the process has been applied k times, with $1 \leq k \leq n-1$. It is assumed that the induction hypotheses at step k are the following:

$$(H(k;a)) \quad D_{k-1} \text{ is a subset of } D \text{ satisfying}$$

$$\alpha(D_{k-1}, K_j) \leq (2k-1)\varepsilon^{2n-k+1} \quad \text{for all } 1 \leq j \leq n-k+1;$$

$$(H(k;b)) \quad P_{n-k+1} \subset S \text{ is an } (\varepsilon, K_{n-k+1})\text{-filling pattern for } D_{k-1};$$

$$(H(k;c)) \quad \text{setting}$$

$$D_k := D_{k-1} \setminus K_{n-k+1} P_{n-k+1},$$

we have

$$|D_k| \leq |D| \prod_{0 \leq i \leq k-1} (1 - \varepsilon(1 - (2i+1)\varepsilon^{2n-i})).$$

Note that these induction hypotheses are satisfied for $k=1$ by Step 1.

Let us pass to Step $k+1$.

Step $k+1$. If $|D_k| \leq \varepsilon|D_{k-1}|$ and hence $|D_k| \leq \varepsilon|D|$, then we take $P_j = \emptyset$ for all $1 \leq j \leq n-k$ and stop the process.

Otherwise, we have $|D_k| > \varepsilon|D_{k-1}|$. Let us estimate from above, for all $1 \leq j \leq n-k$, the relative amenability constants $\alpha(D_k, K_j)$.

Let $1 \leq j \leq n-k$.

If $P_{n-k+1} = \emptyset$, then $D_k = D_{k-1}$ and therefore

$$\begin{aligned} \alpha(D_k, K_j) &= \alpha(D_{k-1}, K_j) \\ &\leq (2k-1)\varepsilon^{2n-k+1} && \text{(by our induction hypothesis (H(k;a)))} \\ &\leq (2k+1)\varepsilon^{2n-k} && \text{(since } 0 < \varepsilon < 1\text{).} \end{aligned}$$

Suppose now that $P_{n-k+1} \neq \emptyset$. Then we can apply Lemma 3.4 with $\Omega := D_{k-1}$ and $A := K_{n-k+1}P_{n-k+1}$. This gives us

$$(3.11) \quad \alpha(D_k, K_j) = \alpha(D_{k-1} \setminus K_{n-k+1}P_{n-k+1}, K_j) \leq \frac{\alpha(D_{k-1}, K_j) + |K_j|\alpha(K_{n-k+1}P_{n-k+1}, K_j)}{\varepsilon}.$$

Proposition 2.1.(vi) and condition (3.8) imply that, for all $s \in S$,

$$\alpha(K_{n-k+1}s, K_j) = \alpha(K_{n-k+1}, K_j) \leq \frac{\varepsilon^{2n}}{|K_j|}.$$

Since the family $(K_{n-k+1}s)_{s \in P_{n-k+1}}$ is ε -disjoint, the preceding inequality together with Lemma 3.3 give us

$$\alpha(K_{n-k+1}P_{n-k+1}, K_j) = \alpha\left(\bigcup_{s \in P_{n-k+1}} K_{n-k+1}s, K_j\right) \leq \frac{\varepsilon^{2n}}{(1-\varepsilon)|K_j|}.$$

From inequality (3.11) and the induction hypothesis (H(k;a)), we deduce that

$$\alpha(D_k, K_j) \leq \frac{(2k-1)\varepsilon^{2n-k+1}}{\varepsilon} + \frac{\varepsilon^{2n}}{(1-\varepsilon)\varepsilon} \leq (2k+1)\varepsilon^{2n-k}$$

(for the second inequality, observe that $1/(1-\varepsilon) \leq 2$ since $0 < \varepsilon \leq 1/2$).

This shows (H(k+1;a)).

Using Lemma 3.7 with $\Omega := D_k$ and $K := K_{n-k}$, we can find a finite subset $P_{n-k} \subset S$ such that P_{n-k} is an (ε, K_{n-k}) -filling pattern for D_k , thus yielding (H(k+1;b)), and satisfying

$$(3.12) \quad |K_{n-k}P_{n-k}| \geq \varepsilon (1 - \alpha(D_k, K_{n-k}))|D_k| \geq \varepsilon (1 - (2k+1)\varepsilon^{2n-k})|D_k|.$$

Setting

$$D_{k+1} := D_k \setminus K_{n-k}P_{n-k},$$

we deduce from (3.12) that

$$|D_{k+1}| \leq |D_k|(1 - \varepsilon(1 - (2k+1)\varepsilon^{2n-k})).$$

Together with the inequality of the induction hypothesis (H(k;c)), this yields

$$|D_{k+1}| \leq |D| \prod_{0 \leq i \leq k} (1 - \varepsilon(1 - (2i+1)\varepsilon^{2n-i})).$$

Thus condition (H(k+1;c)) is also satisfied. This finishes the construction of Step $k+1$ and proves the induction step.

Now, suppose that this process continues until Step n . Using (H(k;c)) for $k = n$, we obtain

$$(3.13) \quad |D_n| \leq |D| \prod_{0 \leq i \leq n-1} (1 - \varepsilon(1 - (2i+1)\varepsilon^{2n-i})).$$

We will show that for $n \geq n_0$, with $n_0 = n_0(\varepsilon)$ only depending on ε , we get $|D_n| \leq \varepsilon|D|$.

As $(2i+1)\varepsilon^{2n-i} \leq (2n+1)\varepsilon^{n+1}$ for all $0 \leq i \leq n-1$, we deduce from (3.13), that

$$(3.14) \quad |D_n| \leq |D| \left(1 - \varepsilon(1 - (2n+1)\varepsilon^{n+1})\right)^n.$$

Since $\lim_{r \rightarrow +\infty} (2r+1)\varepsilon^{r+1} = 0$ and $\lim_{r \rightarrow +\infty} (1 - \frac{\varepsilon}{2})^r = 0$, both monotonically for large r , we can find an integer $n_0 = n_0(\varepsilon) \geq 1$ such that for all $r \geq n_0$, we have both $(2r+1)\varepsilon^{r+1} \leq \frac{1}{2}$ and $(1 - \frac{\varepsilon}{2})^r \leq \varepsilon$. Now, if $n \geq n_0$, using inequality (3.14) we deduce

$$|D_n| \leq |D| \left(1 - \frac{\varepsilon}{2}\right)^n \leq \varepsilon |D|.$$

This finishes the proof of the theorem. \square

4. PROOF OF THE MAIN RESULT

In this section, we give the proof of Theorem 1.1.

So let S be a cancellative left-amenable semigroup and let $h: \mathcal{P}_{fin}(S) \rightarrow \mathbb{R}$ be a real-valued map satisfying conditions (H1), (H2) and (H3).

First observe that by taking $A = B$ in condition (H1), we get $h(A) \leq 2h(A)$ and hence

$$(4.1) \quad h(A) \geq 0 \quad \text{for all } A \in \mathcal{P}_{fin}(S).$$

On the other hand, we deduce from (H1) that

$$h(A) = h\left(\bigcup_{s \in A} \{s\}\right) \leq \sum_{s \in A} h(\{s\})$$

so that, by using (H3), we get

$$(4.2) \quad h(A) \leq M|A| \quad \text{for all } A \in \mathcal{P}_{fin}(S).$$

Let $(F_i)_{i \in I}$ be a left-Følner net for S . By Proposition 2.4, we have

$$(4.3) \quad \lim_i \alpha(F_i, K) = 0 \quad \text{for every finite subset } K \subset S.$$

Consider the quantity

$$(4.4) \quad \lambda := \liminf_i \frac{h(F_i)}{|F_i|}.$$

Note that $0 \leq \lambda \leq M$ by (4.1) and (4.2).

Recall that one says that a finite sequence $(K_j)_{1 \leq j \leq n}$ is *extracted* from the net $(F_i)_{i \in I}$ if there are indices

$$i_1 < i_2 < \dots < i_n$$

in I such that $K_j = F_{i_j}$ for all $1 \leq j \leq n$.

Let $\varepsilon > 0$ and let n be a positive integer. By (4.3) and (4.4), it is clear that we can find, using induction on n , a finite sequence $(K_j)_{1 \leq j \leq n}$ extracted from the net $(F_i)_{i \in I}$ such that:

$$\alpha(K_k, K_j) \leq \frac{\varepsilon^{2n}}{|K_j|} \quad \text{for all } 1 \leq j < k \leq n$$

and

$$(4.5) \quad \frac{h(K_j)}{|K_j|} \leq \lambda + \varepsilon \quad \text{for all } 1 \leq j \leq n.$$

Suppose now that $0 < \varepsilon \leq \frac{1}{2}$ and that $n \geq n_0$, where $n_0 = n_0(\varepsilon)$ is as in Theorem 3.8.

Let $D \subset S$ be a non-empty finite subset satisfying $\alpha(D, K_j) \leq \varepsilon^{2n}$ for all $1 \leq j \leq n$.

By Theorem 3.8, we can find a sequence $(P_j)_{1 \leq j \leq n}$ of finite subsets of S satisfying the following conditions:

- (T1) the set P_j is an (ε, K_j) -filling pattern for D for every $1 \leq j \leq n$;
- (T2) the subsets $K_j P_j \subset D$, $1 \leq j \leq n$, are pairwise disjoint;
- (T3) the subset $D' \subset D$ defined by

$$D' := D \setminus \bigcup_{1 \leq j \leq n} K_j P_j$$

is such that $|D'| \leq \varepsilon |D|$.

We then have

$$D = \bigcup_{1 \leq j \leq n} K_j P_j \cup D'.$$

By applying the subadditivity property (H1) of h , it follows that

$$(4.6) \quad h(D) \leq \sum_{1 \leq j \leq n} h(K_j P_j) + h(D').$$

As $|D'| \leq \varepsilon |D|$ by (T3), we deduce from (4.2) that

$$(4.7) \quad h(D') \leq M\varepsilon |D|.$$

On the other hand, for all $1 \leq j \leq n$, we have

$$\begin{aligned} h(K_j P_j) &= h\left(\bigcup_{s \in P_j} K_j s\right) \\ &\leq \sum_{s \in P_j} h(K_j s) && \text{(by the subadditivity property (H1))} \\ &\leq \sum_{s \in P_j} h(K_j) && \text{(by the right-subinvariance property (H2))} \\ &= \sum_{s \in P_j} \frac{h(K_j)}{|K_j|} |K_j s| && \text{(since } |K_j| = |K_j s| \text{ by right-cancellability of } s) \\ &\leq (\lambda + \varepsilon) \sum_{s \in P_j} |K_j s| && \text{(by (4.5)).} \end{aligned}$$

As the family $(K_j s)_{s \in P_j}$ is ε -disjoint by (T1), we then deduce from Lemma 3.2 that

$$h(K_j P_j) \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} \left| \bigcup_{s \in P_j} K_j s \right| = \frac{\lambda + \varepsilon}{1 - \varepsilon} |K_j P_j|.$$

This implies

$$\sum_{1 \leq j \leq n} h(K_j P_j) \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} \sum_{1 \leq j \leq n} |K_j P_j|$$

and hence

$$(4.8) \quad \sum_{1 \leq j \leq n} h(K_j P_j) \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} |D|,$$

since the sets $K_j P_j$, $1 \leq j \leq n$, are pairwise disjoint subsets of D by (T2).

From (4.6), (4.7), and (4.8), we deduce that

$$(4.9) \quad \frac{h(D)}{|D|} \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} + M\varepsilon.$$

By (4.3), we can find $i_0 \in I$ such that, for all $i \geq i_0$,

$$\alpha(F_i, K_j) \leq \varepsilon^{2n} \quad \text{for all } 1 \leq j \leq n.$$

Hence, by replacing D by F_i for $i \geq i_0$ in inequality (4.9), we obtain

$$\frac{h(F_i)}{|F_i|} \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} + M\varepsilon.$$

This implies

$$\limsup_i \frac{h(F_i)}{|F_i|} \leq \frac{\lambda + \varepsilon}{1 - \varepsilon} + M\varepsilon.$$

Since the latter inequality is satisfied for all $\varepsilon \in (0, \frac{1}{2}]$, taking the limit as ε tends to 0, we obtain

$$\limsup_i \frac{h(F_i)}{|F_i|} \leq \lambda = \liminf_i \frac{h(F_i)}{|F_i|}.$$

This shows that (4.4) is indeed a true limit.

It only remains to show that $\lambda = \lim_i \frac{h(F_i)}{|F_i|}$ does not depend on the choice of the left-Følner net $(F_i)_{i \in I}$. So suppose that $(G_j)_{j \in J}$ is another left-Følner net for S and let $\nu = \lim_j \frac{h(G_j)}{|G_j|}$.

Take disjoint copies I' and J' of the sets I and J , i.e., sets I' and J' with $I \cap I' = \emptyset$ and $J \cap J' = \emptyset$ together with bijective maps $\varphi: I \rightarrow I'$ and $\psi: J \rightarrow J'$. Consider the set $T = (I \times J) \cup (I' \times J')$ with the partial ordering defined as follows. Given $t_1, t_2 \in T$, we

write $t_1 \leq t_2$ if and only if there exist indices $i_1, i_2 \in I$ and $j_1, j_2 \in J$ such that $i_1 \leq i_2$, $j_1 \leq j_2$, and

$$(t_1 = (i_1, j_1) \text{ or } t_1 = (\varphi(i_1), \psi(j_1))) \text{ and } (t_2 = (i_2, j_2) \text{ or } t_2 = (\varphi(i_2), \psi(j_2))).$$

Observe that (T, \leq) is a directed set since (I, \leq) and (J, \leq) are directed sets. Now we define a net $(H_t)_{t \in T}$ of non-empty finite subsets of S by setting

$$H_t = \begin{cases} F_i & \text{if } t = (i, j) \in I \times J, \\ G_j & \text{if } t = (\varphi(i), \psi(j)) \in I' \times J'. \end{cases}$$

Clearly $(H_t)_{t \in T}$ is a left-Følner net for S . By the first part of the proof, the net $\left(\frac{h(H_t)}{|H_t|}\right)_{t \in T}$ converges to some $\tau \geq 0$. Using the fact that for every t_1 in T , there exists t_2 in $I \times J$ (resp. in $I' \times J'$) such that $t_1 \leq t_2$, we conclude that $\tau = \lambda = \nu$. This completes the proof of Theorem 1.1.

5. APPLICATIONS TO DYNAMICAL SYSTEMS

Topological entropy. (cf. [1]) Let X be a compact topological space.

An *open cover* of X is a family of open subsets of X whose union is X . Let $\mathcal{U} = (U_j)_{j \in J}$ and $\mathcal{V} = (V_k)_{k \in K}$ be two open covers of X . One says that \mathcal{V} is *finer* than \mathcal{U} , and one writes $\mathcal{V} \succ \mathcal{U}$, if, for each $k \in K$, there exists $j \in J$ such that $V_k \subset U_j$. One says that \mathcal{V} is a *subcover* of \mathcal{U} if $K \subset J$ and $V_k = U_k$ for all $k \in K$. One writes $\mathcal{U} \cong \mathcal{V}$ if $\{U_j : j \in J\} = \{V_k : k \in K\}$, that is, if the open subsets of X appearing in \mathcal{U} and \mathcal{V} are the same (as soon as we forget that they are indexed).

The *join* of \mathcal{U} and \mathcal{V} is the open cover $\mathcal{U} \vee \mathcal{V}$ of x defined by $\mathcal{U} \vee \mathcal{V} := (U_j \cap V_k)_{(j,k) \in J \times K}$. If $f: X \rightarrow X$ is a continuous map, the *pullback* of \mathcal{U} by f is the open cover $f^{-1}(\mathcal{U})$ of X defined by $f^{-1}(\mathcal{U}) := (f^{-1}(U_j))_{j \in J}$.

Since X is compact, every open cover of X admits a finite subcover. Given an open cover \mathcal{U} of X , let $N(\mathcal{U})$ denote the smallest integer $n \geq 0$ such that \mathcal{U} admits a subcover of cardinality n .

Lemma 5.1. *Let X be a compact space. Let $\mathcal{U} = (U_j)_{j \in J}$ and $\mathcal{V} = (V_k)_{k \in K}$ be two open covers of X . Then one has*

- (i) $N(\mathcal{U} \vee \mathcal{V}) \leq N(\mathcal{U})N(\mathcal{V})$;
- (ii) if $\mathcal{V} \succ \mathcal{U}$ then $N(\mathcal{V}) \geq N(\mathcal{U})$;
- (iii) if $\mathcal{U} \cong \mathcal{V}$ then $N(\mathcal{U}) = N(\mathcal{V})$;
- (iv) if $f: X \rightarrow X$ is a continuous map then $N(f^{-1}(\mathcal{U})) \leq N(\mathcal{U})$.

Proof. These properties are all obvious (see for example [1]). □

Now suppose that the compact space X is endowed with a continuous action of a semi-group S . This means that we are given a map $S \times X \rightarrow X$, $(s, x) \mapsto sx$, satisfying the following conditions: (1) one has $s_1(s_2x) = (s_1s_2)x$ for all $s_1, s_2 \in S$ and $x \in X$; (2) the map $T_s: X \rightarrow X$ defined by $T_s(x) := sx$ is continuous for all $s \in S$.

Let \mathcal{U} be an open cover of X . Consider the map $h_{\mathcal{U}}: \mathcal{P}_{fin}(S) \rightarrow \mathbb{R}$ defined by

$$(5.1) \quad h_{\mathcal{U}}(A) := \log N(\mathcal{U}_A),$$

where

$$(5.2) \quad \mathcal{U}_A := \bigvee_{s \in A} T_s^{-1}(\mathcal{U}).$$

(By convention, $\mathcal{U}_{\emptyset} = \{X\}$ so that $h_{\mathcal{U}}(\emptyset) = 0$.)

Proposition 5.2. *Let X be a compact space equipped with a continuous action of a semi-group S and let \mathcal{U} be an open cover of X . Then the map $h_{\mathcal{U}}: \mathcal{P}_{fin}(S) \rightarrow \mathbb{R}$ defined by (5.1) is non-decreasing, subadditive, right-subinvariant, and uniformly bounded on singletons.*

Proof. Let A and B be finite subsets of S .

If $A \subset B$, then \mathcal{U}_B is finer than \mathcal{U}_A . This implies $N(\mathcal{U}_A) \leq N(\mathcal{U}_B)$ by Lemma 5.1.(ii) and hence $h_{\mathcal{U}}(A) \leq h_{\mathcal{U}}(B)$. This shows that $h_{\mathcal{U}}$ is non-decreasing.

Suppose now that A and B are disjoint. Then we have $\mathcal{U}_{A \cup B} = \mathcal{U}_A \vee \mathcal{U}_B$ and hence $N(\mathcal{U}_{A \cup B}) \leq N(\mathcal{U}_A)N(\mathcal{U}_B)$. This implies $h_{\mathcal{U}}(A \cup B) \leq h_{\mathcal{U}}(A) + h_{\mathcal{U}}(B)$.

If A and B are arbitrary subsets of S , we can write

$$\begin{aligned} h_{\mathcal{U}}(A \cup B) &= h_{\mathcal{U}}((A \setminus B) \cup B) \\ &\leq h_{\mathcal{U}}(A \setminus B) + h_{\mathcal{U}}(B) && \text{(since } A \setminus B \text{ and } B \text{ are disjoint)} \\ &\leq h_{\mathcal{U}}(A) + h_{\mathcal{U}}(B) && \text{(since } h \text{ is non-decreasing).} \end{aligned}$$

this shows that $h_{\mathcal{U}}$ is subadditive.

To prove right-subinvariance, we first observe that, for every $s \in S$ and any finite subset A of S , we have

$$\begin{aligned} \mathcal{U}_{As} &= \bigvee_{t \in As} T_t^{-1}(\mathcal{U}) \\ &\cong \bigvee_{a \in A} T_{as}^{-1}(\mathcal{U}) \\ &= \bigvee_{a \in A} (T_a \circ T_s)^{-1}(\mathcal{U}) \\ &= \bigvee_{a \in A} T_s^{-1}(T_a^{-1}(\mathcal{U})) \\ &= T_s^{-1} \left(\bigvee_{a \in A} T_a^{-1}(\mathcal{U}) \right) \\ &= T_s^{-1}(\mathcal{U}_A). \end{aligned}$$

We then deduce that

$$h_{\mathcal{U}}(As) = \log N(\mathcal{U}_{As}) = \log N(T_s^{-1}(\mathcal{U}_A)) \leq \log N(\mathcal{U}_A) = h_{\mathcal{U}}(A)$$

by using assertions (iii) and (iv) in Lemma 5.1. This shows that $h_{\mathcal{U}}$ is right-subinvariant.

Finally, for all $s \in S$, we have

$$h_{\mathcal{U}}(\{s\}) = \log N(T_s^{-1}(\mathcal{U})) \leq \log N(\mathcal{U})$$

by Lemma 5.1.(iv). It follows that $h_{\mathcal{U}}$ is uniformly bounded on singletons. \square

From Proposition 5.2 and Theorem 1.1, we deduce the following result.

Theorem 5.3. *Let X be a compact space equipped with a continuous action of a cancellative left-amenable semigroup S and let \mathcal{U} be an open cover of X . Then, for every left-Følner net $(F_i)_{i \in I}$ of S , the limit*

$$\eta_{\mathcal{U}} := \lim_i \frac{h_{\mathcal{U}}(F_i)}{|F_i|}$$

exists and is finite. Moreover, $\eta_{\mathcal{U}}$ does not depend on the choice of the left-Følner net $(F_i)_{i \in I}$.

The quantity $0 \leq \eta \leq +\infty$ defined by $\eta := \sup_{\mathcal{U}} \eta_{\mathcal{U}}$, where \mathcal{U} runs over all open covers of X , is the *topological entropy* of the continuous dynamical system (X, S) .

Topological mean dimension. (cf. [10], [15], [3], [2]) Let X be a compact metrizable space.

Let $\mathcal{U} = (U_j)_{j \in J}$ be a finite open cover of X . The *local order* of \mathcal{U} at a point $x \in X$ is the integer $\text{ord}(\mathcal{U}, x) := 1 + m(\mathcal{U}, x)$, where $m(\mathcal{U}, x)$ is the number of indices $j \in J$ such that $x \in U_j$. The *order* of \mathcal{U} is the integer $\text{ord}(\mathcal{U}) := \max_{x \in X} \text{ord}(\mathcal{U}, x)$. Define the integer $D(\mathcal{U})$ by $D(\mathcal{U}) := \min_{\mathcal{V}} \text{ord}(\mathcal{V})$, where \mathcal{V} runs over all finite open covers of X such that $\mathcal{V} \succ \mathcal{U}$. The quantity $0 \leq \dim(X) \leq +\infty$ defined by $\dim(X) := \sup_{\mathcal{U}} D(\mathcal{U})$, where \mathcal{U} runs over all finite open covers of X , is the *topological dimension* of X (cf. [11]).

Lemma 5.4. *Let X be a compact metrizable space. Let $\mathcal{U} = (U_j)_{j \in J}$ and $\mathcal{V} = (V_k)_{k \in K}$ be two finite open covers of X . Then one has*

- (i) $D(\mathcal{U} \vee \mathcal{V}) \leq D(\mathcal{U}) + D(\mathcal{V})$;
- (ii) if $\mathcal{V} \succ \mathcal{U}$ then $D(\mathcal{V}) \geq D(\mathcal{U})$;
- (iii) if $\mathcal{U} \cong \mathcal{V}$ then $D(\mathcal{U}) = D(\mathcal{V})$;
- (iv) if $f: X \rightarrow X$ is a continuous map then $D(f^{-1}(\mathcal{U})) \leq D(\mathcal{U})$.

Proof. See for example [15], [3], or [2]. \square

Let X be a compact metrizable space equipped with a continuous action of a semigroup S . Let \mathcal{U} be a finite open cover of X . Consider the map $h_{\mathcal{U}}^{\dim}: \mathcal{P}_{fin}(S) \rightarrow \mathbb{R}$ defined by

$$(5.3) \quad h_{\mathcal{U}}^{\dim}(A) := D(\mathcal{U}_A),$$

where \mathcal{U}_A is defined by (5.2).

Proposition 5.5. *Let X be a compact metrizable space equipped with a continuous action of a semigroup S and let \mathcal{U} be a finite open cover of X . Then the map $h_{\mathcal{U}}^{\dim}: \mathcal{P}_{fin}(S) \rightarrow \mathbb{R}$ defined by (5.3) is non-decreasing, subadditive, right-subinvariant, and uniformly bounded on singletons.*

Proof. Mutatis mutandis, the proof is that of Proposition 5.2 with Lemma 5.4 replacing Lemma 5.1. \square

From Proposition 5.5 and Theorem 1.1, we deduce the following result.

Theorem 5.6. *Let X be a compact metrizable space equipped with a continuous action of a cancellative left-amenable semigroup S and let \mathcal{U} be a finite open cover of X . Then, for every left-Følner net $(F_i)_{i \in I}$ of S , the limit*

$$\eta_{\mathcal{U}}^{\dim} := \lim_i \frac{h_{\mathcal{U}}^{\dim}(F_i)}{|F_i|}$$

exists and is finite. Moreover, $\eta_{\mathcal{U}}^{\dim}$ does not depend on the choice of the left-Følner net $(F_i)_{i \in I}$.

The quantity $0 \leq \eta^{\dim} \leq +\infty$ defined by $\eta^{\dim} := \sup_{\mathcal{U}} \eta_{\mathcal{U}}^{\dim}$, where \mathcal{U} runs over all finite open covers of X , is the *topological mean dimension* of the continuous dynamical system (X, S) .

Measure-theoretic entropy. (cf. [13], [19], [12]) Let $X = (X, \mathcal{B}, p)$ be a probability space.

A *finite measurable partition* of X is a finite family $\mathcal{U} = (U_j)_{j \in J}$ of pairwise disjoint measurable subsets of X whose union is X (here, equalities for subsets of X are understood to hold up to null-measure sets). The join operation \vee , as well as the relations \succ and \cong , can also be defined for finite measurable partitions. Moreover, if $T: X \rightarrow X$ is a measurable map and $\mathcal{U} = (U_j)_{j \in J}$ is a finite measurable partition of X , then $T^{-1}(\mathcal{U}) := (T^{-1}(U_j))_{j \in J}$ is also a finite measurable partition of X .

If $\mathcal{U} = (U_j)_{j \in J}$ is a finite measurable partition of X , we define the real number $E(\mathcal{U}) \geq 0$ by

$$E(\mathcal{U}) := - \sum_{j \in J} p(U_j) \log p(U_j),$$

with the usual convention $0 \log 0 = 0$.

A measurable map $T: X \rightarrow X$ is said to be *measure-preserving* if $p(T^{-1}(B)) = p(B)$ for all $B \in \mathcal{B}$.

Lemma 5.7. *Let (X, \mathcal{B}, p) be a probability space. Let $\mathcal{U} = (U_j)_{j \in J}$ and $\mathcal{V} = (V_k)_{k \in K}$ be two finite measurable partitions of X . Then one has*

- (i) $E(\mathcal{U} \vee \mathcal{V}) \leq E(\mathcal{U}) + E(\mathcal{V})$;
- (ii) if $\mathcal{V} \succ \mathcal{U}$ then $E(\mathcal{V}) \geq E(\mathcal{U})$;
- (iii) if $\mathcal{U} \cong \mathcal{V}$ then $E(\mathcal{U}) = E(\mathcal{V})$;
- (iv) if $T: X \rightarrow X$ is a measure-preserving map then $E(T^{-1}(\mathcal{U})) = E(\mathcal{U})$.

Proof. See for example [12, Section 4.3]. \square

Let (X, \mathcal{B}, p) be a probability space. Suppose that X is equipped with a *measure-preserving action* of a semigroup S , that is, a family of measure-preserving maps $T_s: X \rightarrow X$, $s \in S$, such that

$$T_{s_1} \circ T_{s_2} = T_{s_1 s_2} \quad p - \text{a.e.}$$

for all $s_1, s_2 \in S$.

Let \mathcal{U} be a finite measurable partition of X . Consider the map $h_{\mathcal{U}}^{\text{KS}}: \mathcal{P}_{\text{fin}}(S) \rightarrow \mathbb{R}$ defined by

$$(5.4) \quad h_{\mathcal{U}}^{\text{KS}}(A) := E(\mathcal{U}_A),$$

where \mathcal{U}_A is defined by (5.2).

Proposition 5.8. *Let (X, \mathcal{B}, p) be a probability space equipped with a measure-preserving action of a semigroup S and let \mathcal{U} be a finite measurable partition of X . Then the map $h_{\mathcal{U}}^{\text{KS}}: \mathcal{P}_{\text{fin}}(S) \rightarrow \mathbb{R}$ defined by (5.4) is non-decreasing, subadditive, right-invariant, and uniformly bounded on singletons.*

Proof. Mutatis mutandis, the proof is that of Proposition 5.2 with Lemma 5.7 replacing Lemma 5.1. Note that $h_{\mathcal{U}}^{\text{KS}}$ is indeed right-invariant since in Lemma 5.7.(iv) an equality holds. \square

From Proposition 5.8 and Theorem 1.1, we deduce the following result.

Theorem 5.9. *Let (X, \mathcal{B}, p) be a probability space equipped with a measure-preserving action of a cancellative left-amenable semigroup S and let \mathcal{U} be a finite measurable partition of X . Then, for every left-Følner net $(F_i)_{i \in I}$ of S , the limit*

$$\eta_{\mathcal{U}}^{\text{KS}} := \lim_i \frac{h_{\mathcal{U}}^{\text{KS}}(F_i)}{|F_i|}$$

exists and is finite. Moreover, $\eta_{\mathcal{U}}^{\text{KS}}$ does not depend on the choice of the left-Følner net $(F_i)_{i \in I}$.

The quantity $0 \leq \eta^{\text{KS}} \leq +\infty$ defined by $\eta^{\text{KS}} := \sup_{\mathcal{U}} \eta_{\mathcal{U}}^{\text{KS}}$, where \mathcal{U} runs over all finite measurable partitions of X , is the *measure-theoretic entropy*, or *Kolmogoroff-Sinai entropy*, of the measure-preserving dynamical system (X, S) .

REFERENCES

- [1] R. L. ADLER, A. G. KONHEIM, AND M. H. MCANDREW, *Topological entropy*, Trans. Amer. Math. Soc., 114 (1965), pp. 309–319.
- [2] M. COORNAERT, *Dimension topologique et systèmes dynamiques*, vol. 14 of Cours Spécialisés [Specialized Courses], Société Mathématique de France, Paris, 2005.
- [3] M. COORNAERT AND F. KRIEGER, *Mean topological dimension for actions of discrete amenable groups*, Discrete Contin. Dyn. Syst., 13 (2005), pp. 779–793.
- [4] M. M. DAY, *Means for the bounded functions and ergodicity of the bounded representations of semi-groups*, Trans. Amer. Math. Soc., 69 (1950), pp. 276–291.
- [5] ———, *Amenable semigroups*, Illinois J. Math., 1 (1957), pp. 509–544.
- [6] ———, *Semigroups and amenability*, in Semigroups (Proc. Sympos., Wayne State Univ., Detroit, Mich., 1968), Academic Press, New York, 1969, pp. 5–53.
- [7] M. FEKETE, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z., 17 (1923), pp. 228–249.
- [8] E. FÖLNER, *On groups with full Banach mean value*, Math. Scand., 3 (1955), pp. 243–254.
- [9] A. H. FREY, JR., *Studies on amenable semigroups*, ProQuest LLC, Ann Arbor, MI, 1960. Thesis (Ph.D.)—University of Washington.

- [10] M. GROMOV, *Topological invariants of dynamical systems and spaces of holomorphic maps. I*, Math. Phys. Anal. Geom., 2 (1999), pp. 323–415.
- [11] W. HUREWICZ AND H. WALLMAN, *Dimension Theory*, Princeton Mathematical Series, v. 4, Princeton University Press, Princeton, N. J., 1941.
- [12] A. KATOK AND B. HASSELBLATT, *Introduction to the modern theory of dynamical systems*, vol. 54 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.
- [13] A. N. KOLMOGOROV, *A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces*, Dokl. Akad. Nauk SSSR (N.S.), 119 (1958), pp. 861–864.
- [14] F. KRIEGER, *Le lemme d’Ornstein-Weiss d’après Gromov*, in Dynamics, ergodic theory, and geometry, vol. 54 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 2007, pp. 99–111.
- [15] E. LINDENSTRAUSS AND B. WEISS, *Mean topological dimension*, Israel J. Math., 115 (2000), pp. 1–24.
- [16] I. NAMIOKA, *Følner’s conditions for amenable semi-groups*, Math. Scand., 15 (1964), pp. 18–28.
- [17] J. VON NEUMANN, *Zur Allgemeine Theorie der Massen*, Fund. Math., 13 (1929), pp. 73–116.
- [18] D. S. ORNSTEIN AND B. WEISS, *Entropy and isomorphism theorems for actions of amenable groups*, J. Analyse Math., 48 (1987), pp. 1–141.
- [19] YA. G. SINAI, *On the concept of entropy for a dynamic system*, Dokl. Akad. Nauk SSSR, 124 (1959), pp. 768–771.

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